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THE METHOD OF AVERAGING IN DYNAMIC PROBLEMS OF THE THEORY OF THE ELASTICITY OF STRUCTURALLY INHOMOGENEOUS MEDIA*

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A modification of the method of averaging to construct approximate solutions of the dynamic equations of the theory of the elasticity of an inhomogeneous medium as the synthesis of successive approximation with a probabilistic approach is considered. To find the zero-th approximation we assume the material coefficients of the medium to be random functions of the spatial coordinates, one form of which is the elastic moduli and the density of this specific medium. In the zero-th approximation the inhomogeneous medium under consideration is described by an effective medium possessing spatial dispersion. The first approximation takes account of the influence of structural singularities of this specific realization on the displacement field configuration of the propagating waves. Error estimates are given for the zero-th approximation.

The dynamics of inhomogeneous media is described by equations with variable coefficients whose approximate methods of solution have been developed intensively. Different modifications of the averaging method were examined for example, in /1-5/. The method of using the averaging method for discretely laminar structures is described in /1/. An analogous scheme is developed in this paper for the dynamic equations of the theory of the elasticity of inhomogeneous media.

1. The relation between the stress tensor σ_{ij} and the strain tensor e_{ij} in a linear inhomogeneous medium is given by the generalized Hooke's law

$$\sigma_{ij} = \lambda_{ijkl}(x)\, e_{kl},\ e_{kl} = {}^1/_2\,(u_{k,l} + u_{l,k}) \tag{1.1}$$

where λ_{ijkl} is the tensor of the elastic coefficients whose components depend on the spatial coordinates x_i $(i = 1, 2, 3)$.

Harmonic wave propagation is described by the equations

$$(\lambda_{ijkl}(x)\, u_{k,l})_{,j} + \omega^2 \rho(x)\, u_i = 0 \tag{1.2}$$

Here $\rho(x)$ is the density of the medium and u_i is the displacement vector. The boundary conditions or the conditions at infinity are deterministic and posed in the ordinary way for each specific problem.

In direct problems $\lambda_{ijkl}(x)$, $\rho(x)$ are specific functions of x_i. We assume that the data $\lambda_{ijkl}(x)$, $\rho(x)$ are samples of random fields $\Lambda_{ijkl}(x)$, $P(x)$ given by their moments. Later, we shall denote the random functions and samples of them by identical letters as is customary in wave propagation theory and does not result in errors. Multiplication of $\lambda_{ijkl}(x), \rho(x)$ by a certain field function $f(x)$ will be considered as the action of the operators $\lambda_{ijkl}(x)$, $\rho(x)$ on $f(x)$. We write the symbolic operator representation

$$\lambda_{ijkl} = \Lambda_{ijkl}^* + \Lambda_{ijkl}',\ \rho = \rho^* + \rho',\ \Lambda' f = \lambda f - \Lambda^* f \tag{1.3}$$

where Λ_{ijkl}^*, ρ^* are the effective elasticity and density operators introduced by the relationships

$$\langle \sigma_{ij} \rangle = \langle \lambda_{ijkl} e_{kl} \rangle = \Lambda_{ijkl}^* \langle e_{kl} \rangle,\ \langle \rho u_i \rangle = \rho^* \langle u_i \rangle \tag{1.4}$$

The mean fields $\langle \sigma_{ij} \rangle$, $\langle e_{ij} \rangle$, $\langle u_i \rangle$ are determined for an effective medium characterized by Λ_{ijkl}^*, ρ^*. The representation (1.3) does not result in errors since the operators in all the calculations are considered jointly with the functions to which they are applied.

According to (1.4), we represent (1.2) in the form

$$u_i(x) = \langle u_i(x) \rangle + \sum_{n=1}^{\infty} u_i^{(n)}(x) \tag{1.5}$$

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Substituting (1.5) into (1.2), we obtain

$$(\Lambda^*_{ijkl}u^{(n)}_{k,l})_{,j} + \rho^*\omega^2 u^{(n)}_i = -(\Lambda'_{ijkl}u^{(n-1)}_{k,l})_{,j} - \rho'\omega^2 u^{(n-1)}_i, \tag{1.6}$$
$$n = 0, 1, 2, \ldots, \quad u^{(-1)}_i = 0, \quad u^{(0)}_i = \langle u_i\rangle$$

We will write (1.6) in direct notation, which we will also use later by changing when necessary to a subscript representation

$$L^* u^{(n)} = L' u^{(n-1)}, \quad L_{ik}^* = \nabla_j \Lambda^*_{ijkl}\nabla_l + \rho^*\omega^2\delta_{ik}, \tag{1.7}$$
$$L_{ik}' = \nabla_j \Lambda'_{ijkl}\nabla_l + \omega^2\rho'\delta_{ik}$$

The solution of (1.7) has the form

$$u^{(n)} = -(L^*)^{-1}L'u^{(n-1)} = (-1)^n M^{(n)}\langle u\rangle, \tag{1.8}$$
$$M^{(n)} = (L^*)^{-1}L' \ldots (L^*)^{-1}L'$$

where L^* is an integrodifferential operator and $(L^*)^{-1}$ is the inverse operator to L^*.
Taking (1.8) into account we can write the solution (1.5) in the form

$$u(x) = \langle u(x)\rangle + \sum_{n=1}^{\infty}(-1)^n M^{(n)}\langle u(x)\rangle \tag{1.9}$$

Using Green's tensor $G^*(x, x_1)$ of the effective medium, the operator $(L^*)^{-1}$ can be represented as follows:

$$(L^*)^{-1}f(x) = \int G^*(x, x_1) f(x_1)\, d^3x_1, \; L^*G^*(x, x_1) = \delta(x - x_1) \tag{1.10}$$

Taking account of (1.10), series (1.9) is the scattering series for the problem of the scattering of the field $\langle u\rangle$ by the fluctuations Λ', ρ' of an inhomogeneous medium. Here the n-th term of the series (1.9) corresponds to taking account of n-tuple scattering of the field $\langle u\rangle$ by inhomogeneity fluctuations. For given functions $\lambda(x)$, $\rho(x)$ and known Λ^*, ρ^* (or G^*) the series (1.9) yields the solution of (1.3).
The method of changing from (1.3) to the integral

$$u(x) = \langle u(x)\rangle - \int G^*(x, x_1)[(\Lambda'e)_{,x_1} + \rho'\omega^2 u](x_1)\, d^3x_1 \tag{1.11}$$

and that obtained from it by differentiation with respect to x

$$e(x) = \langle e(x)\rangle - \int G_{,x}^*(x, x_1)[(\Lambda'e)_{,x_1} + \rho'\omega^2 u](x_1)\, d^3x_1 \tag{1.12}$$

are equivalent to the one considered.
To take account of multiple scattering we will change from the variables e, Λ' in (1.12) to E, γ according to the formulas /7/

$$E_{ij} = B_{ijkl}e_{kl}, \quad \gamma_{njst} = \lambda'_{njkl}B^{-1}_{klst}$$
$$B_{imkl} = 2^{-1}(\delta_{ik}\delta_{ml} + \delta_{il}\delta_{mk}) + G^{(s)}_{in,mj}\lambda'_{njkl}$$

where $G^*_{in,mj}(x, x_1) = G^{(r)*}_{in,mj}(x, x_1) + G^{(s)}_{in,mj}\delta(x - x_1)$ ($\gamma_{njst}(x)$ is the tensor of elastic polarizability of the medium).
In the new variables, (1.12) has the form

$$E(x) = \langle e(x)\rangle - \int [G^{(r)*}_{,xx}(x, x_1)(\gamma E)(x_1) + G^{(r)*}_{,x}(x, x_1)\,\omega^2(\rho'u)(x_1)]\, d^3x_1 \tag{1.13}$$

We will write the system of integral Eqs.(1.11) and (1.13) in matrix form. In the usual manner we will change to matrices and vectors /8/

$$(11)\to 1,\ (22)\to 2,\ (33)\to 3,\ (23) = (32)\to 4$$
$$(31) = (13)\to 5,\ (12) = (21)\to 6,\ \alpha, \beta = 1, \ldots, 6$$
$$\lambda_{ijkl} = \lambda_{\alpha\beta},\ \gamma_{ijkl} = \gamma_{\alpha\beta},\ G^*_{ij,m} = g_{\alpha m},\ m = 1, 2, 3$$

where $(ij) \to \alpha$, $(kl) \to \beta$. We will introduce the notation /8/ (the superscript T denotes transposition)

$$\Phi = \mathrm{col}\,\{u_1,\ u_2,\ u_3,\ e_1..e_6\},\ \Psi = \mathrm{col}\,\{u_1,\ u_2,\ u_3,\ E_1, \ldots E_6\}$$

$$\gamma_{\alpha\beta}^{(\rho)} = \begin{Vmatrix} \omega^2\rho' I & O \\ O & \gamma \end{Vmatrix},\quad B_{\alpha\beta}^{(1)} = \begin{Vmatrix} I & O \\ O & B \end{Vmatrix},\quad \lambda_{\alpha\beta} = \begin{Vmatrix} \omega^2\rho' I & O \\ O & \lambda \end{Vmatrix}$$

$$\gamma = \begin{Vmatrix} \gamma_{11} \cdots \gamma_{16} \\ \vdots \qquad \vdots \\ \gamma_{61} \cdots \gamma_{66} \end{Vmatrix},\quad B = \begin{Vmatrix} B_{11} \cdots B_{16} \\ \vdots \qquad \vdots \\ B_{61} \cdots B_{66} \end{Vmatrix},\quad \lambda = \begin{Vmatrix} \lambda_{11} \cdots \lambda_{16} \\ \vdots \qquad \vdots \\ \lambda_{61} \cdots \lambda_{66} \end{Vmatrix}$$

$$g_{\alpha\beta} = \begin{Vmatrix} g_{11} \cdots g_{19} \\ \vdots \\ g_{91} \cdots g_{99} \end{Vmatrix} = \begin{Vmatrix} g_{\alpha\beta}^{(1)} & g_{\alpha\beta}^{(2)T} \\ g_{\alpha\beta}^{(2)} & g_{\alpha\beta}^{(3)} \end{Vmatrix} = \begin{Vmatrix} G_{in}^{*} & G_{in,\,m}^{*T} \\ G_{in,\,m}^{*} & G_{in,\,mj}^{(r)*} \end{Vmatrix}$$

$$g_{\alpha\beta}^{(1)} = G_{in}^{*}\ (\alpha = i = 1,\ 2,\ 3;\ \beta = n = 1,\ 2,\ 3),\ g_{\alpha\beta}^{(2)} = G_{in,\,m}^{*}$$
$$(\alpha = m = 1,\ 2,\ 3;\ (in) \to \beta = 4, \ldots 9),\ g_{\alpha\beta}^{(3)} = G_{in,\,mj}^{(r)*}$$
$$((in) \to \alpha,\ (mj) \to \beta;\ \alpha, \beta = 4, \ldots 9)$$

Here I is the unit and O the zero (6x6) matrix.
The system (1.11) and (1.12) can be reduced to the form

$$\Psi(x) = \langle \Phi(x) \rangle - \int g(x,\ x_1)\, \gamma^{(\rho)}(x_1)\, \Psi(x_1)\, d^3x_1 \tag{1.14}$$

Solving this equation by successive iterations, we represent the result in the form of the series

$$\Psi(x) = \langle \Phi(x) \rangle - \int g(x,\ x_1)\, (\gamma^{(\rho)} \langle \Phi \rangle)(x_1)\, d^3x_1 + \tag{1.15}$$
$$\iint g(x,\ x_1)\, \gamma^{(\rho)}(x_1)\, g(x_1,\ x_2)\, \gamma^{(\rho)}(x_2)\, \langle \Phi(x_2) \rangle\, d^3x_1 d^3x_2 - \ldots$$

Taking the average of (1.15) we obtain

$$\langle \Psi(x) \rangle = \langle \Phi(x) \rangle - \int g(x, x_1)\, \langle \gamma^{(\rho)}(x_1) \rangle\, \langle \Phi(x_1) \rangle\, d^3x_1 +$$
$$\iint g(x,\ x_1)\, g(x_1,\ x_2)\, \langle \gamma^{(\rho)}(x_1)\, \gamma^{(\rho)}(x_2) \rangle\, \langle \Phi(x_2) \rangle\, d^3x_1 d^3x_2 - \ldots$$

It can be confirmed directly that this series is a solution of an equation of Dyson type /6, 7/ for an inhomogeneous elastic medium

$$\langle \Psi(x) \rangle = \langle \Phi(x) \rangle + \iint g(x, x_1)\, Q(x_1, x_2)\, \langle \Psi(x_2) \rangle d^3x_1 d^3x_2 \tag{1.16}$$

The kernel $Q(x_1, x_2)$ of the mass operator Q is a series in the moments $\gamma^{(\rho)}(x)$ $(\langle \gamma^{(\rho)}(x) \rangle = 0)$

$$Q(x_1, x_2) = g(x_1, x_2)\, \langle \gamma^{(\rho)}(x_1)\, \gamma^{(\rho)}(x_2) \rangle - \ldots \tag{1.17}$$

The integral Eq.(1.14) is statistically non-linear; consequently, obtaining equations for the moments of the field Ψ on the basis of this results in an infinite coupled system of equations.

Let us introduce the effective operators Λ^*, ρ^*, $\Gamma^{(\rho)*}$ by the relationships (1.3) and

$$\langle \gamma^{(\rho)} \Psi \rangle = \Gamma^{(\rho)*} \langle \Psi \rangle$$

Taking the average in (1.14), we obtain

$$\langle \Psi(x) \rangle = \langle \Phi(x) \rangle - \int g(x,\ x_1) [\,\Gamma^{(\rho)*} \langle \Psi \rangle]\,(x_1)\, d^3x_1 \tag{1.18}$$

Comparing (1.16) and (1.18), we find

$$\Gamma^{(\rho)*} f(x) = \int \Gamma^{(\rho)*}(x,\ x_1)\, f(x_1)\, d^3x_1 \tag{1.19}$$
$$\Gamma^{(\rho)*}(x,\ x_1) = -g(x,\ x_1)\, \langle \gamma^{(\rho)}(x)\, \gamma^{(\rho)}(x_1) \rangle + \ldots$$

We obtain a formula connecting the operators Λ^*, Γ^*:

$$\Lambda^* = \lambda_0 - \Gamma^* (1 - G^{(s)} \Gamma^*)^{-1} \tag{1.20}$$

from the formulas

$$\Psi = B^{(1)} \Phi,\ \gamma^{(\rho)} = \lambda^{(\rho)} (B^{(1)})^{-1}$$

where λ_0 is the effective static elastic modulus, and $G^{(s)}$ is the singular component of the Green's tensor /7/. For a statistically isotropic inhomogeneous medium

$$T^*f(x) = \int T^*(x - x_1) f(x_1) d^3x_1, \quad T^* = \Gamma^*, \rho^*, \Lambda^* \tag{1.21}$$
$$\Gamma^*(x - x_1) = -G^{(r)}_{,xx}(x - x_1) \langle \gamma(x) \gamma(x_1) \rangle + \ldots$$
$$\rho^*(x - x_1) = -G^{(r)}(x - x_1) \langle \rho'(x) \rho'(x_1) \rangle + \ldots$$
$$\langle \varphi'(x) \varphi'(x_1) \rangle = R\varphi(|x - x_1|), \quad \varphi = \lambda', \rho', \gamma$$
$$\langle \gamma \rangle = 0, \quad \langle \rho'(x) \lambda'(x_1) \rangle = 0$$

It follows from (1.20) and (1.21) that the effective medium takes account of the spatial dispersion.

The operators $\Gamma^*, \rho^*, \Lambda^*$ represent series in the moments $\gamma(x), \rho(x)$ and are written in the general case in the form

$$\Lambda^*f(x) = \int \Lambda^*(x - x_1) f(x_1) d^3x_1, \quad \Lambda^*(x - x_1) = \lambda_0 \delta(x - x_1) + \Lambda^d(x - x_1)$$
$$\Gamma^*_{(x, x_1)} = \sum_{n=2}^{\infty} (-1)^n \langle \gamma^{(n)} \rangle, \quad \rho^*(x, x_1) = \sum_{n=2}^{\infty} (-1)^n \langle \rho^{(n)} \rangle$$
$$\gamma^{(n)} = \int \ldots \int \gamma(x) \gamma(x_1) \ldots \gamma(x_n) G^{(r)}_{,xx}(x, x_1) \ldots G^{(r)}_{,xx}(x_{n-1}, x_n) d^3x_2 \ldots d^3x_n$$
$$\rho^n = \int \ldots \int \rho'(x) \rho'(x_1) \ldots \rho'(x_n) G(x, x_1) \ldots G(x_{n-1}, x_n) d^3x_2 \ldots d^3x_n$$

Here λ_0 are elastic coefficients found from the condition $\langle \gamma_{ijkl}(x) \rangle = 0$.

The elastic coefficients λ_0 for two-phase composites and single-phase polycrystalline materials satisfy algebraic equations analogous to the equations of the selfconsistent field method /7/. It follows from (1.26) that λ_0 are the static local part of the effective elastic models. The term $\Lambda^d(x - x_1)$ takes account of multiple scattering, damping, and dispersion of the wave velocity in an effective medium.

2. After the elasticity operator Λ^* and the density operator ρ^* of the effective medium have been defined in (1.5), we find the zero-th approximation $\langle u \rangle$ and higher approximations $u^{(n)}$ in the series (1.4) according to (1.6) and (1.7). The eigenfunctions of the operators (1.26) are $\exp(iqx)$; consequently, we represent the zero-th approximation $\langle u \rangle$ in the form

$$\langle u(x) \rangle = \sum_n a_n(q_n) \exp(iq^{(n)}x) \tag{2.1}$$

where the wave numbers $q^{(n)}$ are roots of the dispersion equation

$$| q_j \Lambda_{ijkl}(q) q_l - \rho^*(q) \omega^2 \delta_{ik} | = 0 \tag{2.2}$$

In the general case the $q^{(n)}$ are complex

$$q_\alpha^{(n)}(\omega) = \varkappa_\alpha^{(n)}(\omega) + i\delta_\alpha^{(n)}(\omega), \quad \alpha = l, t \tag{2.3}$$

Here $\delta^{(n)}(\omega) = \operatorname{Im} q^{(n)}$ characterizes the damping of the n-th branch because of scattering, the phase velocity $c_{\Phi\alpha}^{(n)} = \omega(\varkappa_\alpha^{(n)})^{-1}$ and group velocity $c_{\Gamma\alpha}^{(n)} = (d\varkappa_\alpha^{(n)}/d\omega)^{-1}$ of the n-th branch are calculated from the known real part of the wave number $\operatorname{Re} q_\alpha^{(n)} = \varkappa_\alpha^{(n)}$.

The distribution of the roots (2.2) and (2.3) in the complex plane q depends on the kind of field correlation functions $\lambda(x), \rho(x)$, characterizing the structure of the medium.

The higher approximations $u^{(n)}$ $(n = 1, 2, \ldots)$ are found from (1.6) and (1.7) according to (1.8) and (1.10) in terms of Green's tensor of the effective medium. An expression is obtained for $G^* = \langle G \rangle$ in /7/

$$\langle G_{ij}(r) \rangle = -\frac{\delta_{ij}}{4\pi r} \sum_{n=0}^{\infty} \frac{q_t^{(n)}}{\chi_t'(q_t^{(n)})} \exp(iq_t^{(n)}r) + \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{1}{4\pi r} \sum_{n=0}^{\infty} \frac{q_\alpha^{(n)}}{\chi_\alpha'(q_\alpha^{(n)})} \exp(iq_\alpha^{(n)}r) \right]_t^l \tag{2.4}$$

for a statistical isotropic homogeneous medium.

Formula (2.4) generalizes the expression for the dynamic Green's tensor G_{ij}^0 of a homogeneous isotropic medium.

Therefore, (1.8), (1.10) and (2.4) enable us in principle, to find all the approximations in the series (1.4) and (1.9). However, the expressions are written in integral form, which makes their practical application difficult.

In many problems one is limited to finding the zero-th approximation, i.e., the approximation of the effective medium /8/. In this case $\langle u \rangle$ is an estimate of the exact solution for a given sample. The error in this estimate for a specific sample is determined by the expression $|u_i - \langle u_i\rangle|$. We can take $\langle (u_i')^2\rangle^{1/2}$, which is the r.m.s deviation of the field component u_i from $\langle u_i\rangle$ as an estimate of the error suitable for a set of samples.

3. We again assume that $\lambda(x), \rho(x)$ are random functions; then

$$u_i' = u_i - \langle u_i\rangle \tag{3.1}$$

are random errors of the approximation of the samples u_i by using $\langle u_i\rangle$.

The correlation tensor

$$V_{ij}(x, x_1) = \langle u_i'(x)\, u_j'(x_1)\rangle \tag{3.2}$$

determines the statistical relations between the errors $u_i'(x)$ $(i = 1, 2, 3)$ at different points. Its diagonal elements, the variances, determine the r.m.s. errors u_i by using the zero-th approximation $\langle u_i\rangle$.

The tensor $V_{ij}(x, x_1)$ can be represented in the form

$$V_{ij}(x, x_1) = K_{ij}(x, x_1) - I_{ij}(x, x_1) \tag{3.3}$$
$$K_{ij}(x, x_1) = \langle u_1(x)\, u_j(x_1)\rangle,\ I_{ij}(x, x_1) = \langle u_i(x)\rangle\langle u_j(x_1)\rangle$$

where $K_{ij}(x, x_1)$ is the coherence function. For $i = j$, $x = x_1$ the quantity I_{ii} determines the intensity of the mean field $\langle u\rangle$, K_{ii} is the mean intensity of the field u and V_{ii} is the fluctuation intensity (errors). Transformation of the energy of the regular part of $\langle u\rangle$ into the fluctuation part u' occurs during wave propagation in an inhomogeneous medium. In the case of the problem of the incidence of a plane wave on an inhomogeneous half-space, K_{ii} remains constant, I_{ii} decreases, and V_{ii} increases. This indicates that the error of the zero-th approximation of $\langle u\rangle$ increases with distance because of scattering, consequently, the higher approximations $u^{(n)}$ must be taken into account.

Writing the series (1.15) at two different points x', x'', multiplying them and taking the average, we obtain a Bethe-Salpeter type equation /6/ for the correlation tensor $V(x', x'')$, which has the following form in a ladder approximation:

$$V(x', x'') = I(x', x'') + \iint \langle G(x', x_1)\rangle\langle \bar{G}(x'', x_2)\rangle \times \tag{3.4}$$
$$R(x_1, x_2)\, V(x_1, x_2)\, d^3x_1 d^3x_2$$

where $R(x_1, x_2)$ is the correlation tensor of the material coefficients of the medium, and the bar denotes the complex conjugate.

Eqs.(1.6) or (1.7) for $n = 0$ or (1.18) for the zero-th approximation $\langle u\rangle$ and the equation for the errors (variances) $V_{ii}(x)$ in (3.4) form a closed system of equations.

4. We will examine the use of the method to find approximate solutions of the equations of the dynamics of a laminated medium described by a one-dimensional model

$$(\lambda(x)\, u_{,x})_{,x} + \omega^2\rho(x)\, u = 0 \tag{4.1}$$

Using the expansions (1.3) and (1.5), we obtain equations for $\langle u\rangle$ and $u^{(1)}$

$$(\Lambda^* \langle u\rangle_{,x})_{,x} + \omega^2\rho^* \langle u\rangle = 0 \tag{4.2}$$

$$(\Lambda^* u^{(1)}_{,x})_{,x} + \omega^2\rho^* u^{(1)} = -[(\Lambda'\langle u\rangle_{,x})_{,x} + \omega^2\rho'\langle u\rangle] \tag{4.3}$$

It is interesting to examine the application of the method to media for which the elastic coefficients can be written explicitly and contain coefficients that can be assumed to be random for the finding the zero-th approximation. Such random functions are called quasideterministic /10/.

We consider a two-component laminated medium

$$\varphi(x) = D_+ + D_-(-1)^{n(-N, x)},\quad D_\pm = {}^1/_2(\varphi_2 \pm \varphi_1) \tag{4.4}$$
$$\varphi(x) = \lambda(x), \rho(x);\ \varphi_i = \lambda_i, \rho_i$$

where λ_i, ρ_i are the elasticity and density coefficients of the i-th component, and $n(-N, x)$ is an entire function governing the number of layer boundaries in the segment $[-N, x]$ possessing the property

$$n(x_1, x_3) = n(x_1, x_2) + n(x_2, x_3),\ x_1 \leqslant x_2 \leqslant x_3 \tag{4.5}$$

To find the zero-th approximation $\langle u\rangle$ we assume that $\lambda(x), \rho(x)$ are random functions. We also assume that $n(-N, x)$ is a stationary Poisson random function possessing the property

(4.5) and the distribution

$$P[n(x_1, x_2) = m] = \frac{\langle n(x_1, x_2)\rangle^m}{m!} \exp[-\langle n(x_1, x_2)\rangle]$$
$$\langle n(x_1, x_2)\rangle = \alpha \mid x_1 - x_2 \mid$$

The numerical characteristics of the random functions $\varphi(x) = \lambda(x)$, $\rho(x)$ have the form

$$\langle\varphi\rangle = D_+, \ \langle\varphi'(x_1)\varphi'(x_2)\rangle = R_\varphi(r) = D_-^2 \exp(-|r|/a) \qquad (4.6)$$
$$a = 1/(2\alpha), \quad r = x_1 - x_2$$

The correlation function (4.6) describes a medium with a completely disordered configuration /8/.

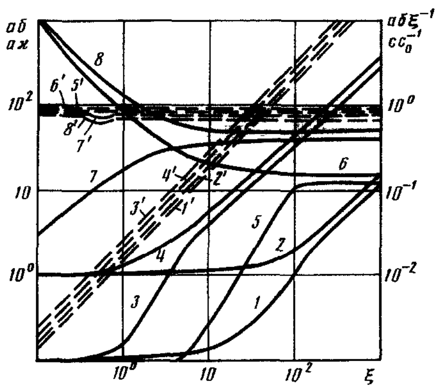
Substituting $\langle u\rangle = a(q)\exp(iqx)$ into (4.2) and considering $\rho^* = \langle\rho\rangle$ we obtain a dispersion equation that we can write in the dimensionless form

$$\alpha^4 + \alpha^2(\beta^2 - \xi^2) - \xi^2\beta(\beta - i\eta\xi) = 0 \qquad (4.7)$$
$$\alpha = aq, \ \xi = ak, \ \eta = {}^1/_2(\pi D), \ \beta = 1 - i\xi$$

The two roots α_1, α_2, located in the first quadrant of the complex plane $\alpha = a\varkappa + ia\delta$ have a physical meaning. The general solution of (4.2) has the form

$$\langle u(x)\rangle = a_1(q)\exp(iq_1x) + a_2(q)\exp(iq_2x) \qquad (4.8)$$

Curves of $a\delta_1$ and $a\delta_2$ (the solid lines) $a\varkappa_1$, $a\varkappa_2$ (the dashed lines) as a function of ξ are shown in the figure by the curves *1, 2 (1′, 2′)* for $\eta = 10^{-3}$ and *3, 4 (3′, 4′)* for $\eta = 1$, respectively, as are also curves of the vibration decrements of the appropriate branches $a\delta_1\xi^{-1}, a\delta_2\xi^{-1}$ (the solid lines) and the velocities $c_1c_0^{-1}$ and $c_2c_0^{-1}$ (the dashed lines) as a function of ξ /curves *5, 6 (5′, 6′)* for $\eta = 10^{-3}$ and *7, 8 (7′, 8′)* for $\eta = 1$/. The parameter ξ characterizes the relationship between the correlation radius a and the wavelength k^{-1}, while the parameter η characterizes the variance of the elastic coefficients of the medium. Substituting (4.8) into (4.3), we find an expression for the first approximation.



To a first approximation (1.9) has the form

$$u = \langle u\rangle + \sum_{m=1}^{2} M_m^{(1)}\langle u_m\rangle \qquad (4.9)$$

$$M_m^{(1)} = \frac{1}{2} iq_m\lambda' \sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} G_j^*(x - x_n)\exp(i(q_m - q_n)x) +$$
$$\lambda' q_m \sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} [\theta(x_{2n} - x)\{I_{2n-1}^{jm}\} + \theta(x - x_{2n})\{I_{2n}^{jm}\} +$$
$$\lambda_m^{(1)} q_m^2 \sum_{j=1}^{2} G_j^*(q_m), \quad \lambda' = \lambda_2 - \lambda_1, \quad \lambda_m^{(1)} = \lambda_1 - \lambda^*(q_m)$$
$$\langle u_m\rangle = a_m(q_m)\exp(iq_mx), \quad \langle u\rangle = \sum_{m=1}^{2}\langle u_m\rangle$$
$$I_{2n-1}^{jm} = \frac{\exp(i(q_m - q_j)x)}{i(q_j - q_m)} \exp(i(q_j - q_m)x_{2n-1})[\exp(i(q_j - q_m)a_{2n}) - 1],$$

$$\theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0, q_j \neq q_m \end{cases}$$

$$I_{2n-1}^{jm} = a_{2n}, \quad q_j = q_m$$

$$I_{2n}^{jm} = -\frac{\exp(iq_jx)}{i(q_j+q_m)} \exp(-i(q_j+q_m)x_{2n-1})[\exp(-i(q_j+q_m)a_{2n}) - 1]$$

$$G_j^*(x - x_n) = \frac{\exp(iq_j|x - x_n|)}{2iq_j}, \quad a_{2n} = x_{2n} - x_{2n-1}$$

$$G_j^*(q_j) = \frac{1}{\lambda^*(q_j)q_j^2 - \langle\rho\rangle\omega^2}, \lambda^*(q) = \frac{\lambda_0}{1-\gamma^*(q)}, \quad \gamma^*(q) = \frac{Dk\beta}{2i(\beta^2+q^2)}$$

It follows from (4.9) and the figure that the zero-th approximation $\langle u\rangle$ damps out over the whole frequency range (for all wavelengths), and the velocity dispersion holds. The structure of the first approximation $u^{(1)}$ is determined by terms with λ' and $\lambda^{(1)}$. To other terms analogous to the last in (4.9) and with $\rho' = \rho_2 - \rho_1$, $\rho^{(1)} = \rho_1 - \rho^*(q)$ are appended to take account of the difference in the densities. The first term for λ' in (4.9) describes the wave field whose radiation sources are the layer boundaries, the second term in λ' describes plane waves in a laminar medium, the third term in $\lambda^{(1)}$ governs the change in amplitude of the field $\langle u\rangle$ (the zero-th approximation) due to the difference in the characteristics of the effective medium from the layer parameters.

Let us consider a medium in which

$$\lambda(x) = \lambda_0 + \lambda_1\cos\theta x + \lambda_2\sin\theta x = \sum_{m=-1}^{1}\lambda^{(m)}\exp(-im\theta x), \quad \rho = \rho_0 \tag{4.10}$$

To find the zero-th approximation $\langle u\rangle$, we assume that λ_1, λ_2 are random independent quantities $\langle\lambda_1\rangle = \langle\lambda_2\rangle = 0$, $\langle\lambda_1\lambda_2\rangle = 0$, $\langle\lambda_1^2\rangle = \langle\lambda_2^2\rangle = D/2$ /10/. Then $\langle\lambda\rangle = \lambda_0$, $\langle\lambda'(x)\lambda'(x+r)\rangle = {}^1/_2 D\cos\theta r$.

The dispersion equation has the form

$$q^2 = k^2[1 + {}^1/_2 iD_\gamma(\delta(k+\theta-q) + \delta(k-\theta-q))] \tag{4.11}$$

It follows from (4.11) that the zero-th approximation for $q \neq k+\theta, k-\theta$ corresponds to a plane undamped wave. For $q = k+\theta, k-\theta$ there are no propagating waves. To a first approximation inclusive, we write the solution of (4.1) in the form

$$\begin{aligned} u(x) &= u_0\exp(ikx) + u_0k^2\lambda'G^*(k)\exp(ikx) + \\ &\frac{1}{2}u_0k(k-\theta)(\lambda_1 - i\lambda_2)G^*(k-\theta)\exp(i(k-\theta)x) + \\ &\frac{1}{2}u_0k(k+\theta)(\lambda_1 + i\lambda_2)G^*(k+\theta)\exp(i(k+\theta)x) = \\ &\sum_{m=-1}^{1} u(q-m\theta)\exp[-i(q-m\theta)x] \end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.1), we obtain the following dispersion equation as in /11/:

$$(\lambda_0q^2 - \rho_0\omega^2)^2 - \lambda^{(1)2}q^4 = 0, \quad \lambda^{(1)} = \lambda_1 + i\lambda_2, \quad \lambda^{(-1)} = \lambda_1 - i\lambda_2 \tag{4.13}$$

from which $\omega^2 = q^2\rho_0^{-1}(\lambda_0 + |\lambda^{(1)}|)$.

The boundaries of the frequency passband are

$$\omega_+^2 = \omega^2(1 + |\lambda^{(1)}|\lambda_0^{-1}), \quad \omega_-^2 = \omega^2(1 - |\lambda^{(1)}|\lambda_0^{-1})$$

The width of the frequency passband is $\Delta\omega = \omega|\lambda^{(1)}|\lambda_0^{-1}$.

The expression for the wave number at the centre of the forbidden band has the form

$$q = \frac{\theta}{2}\left(1 \pm i\frac{|\lambda^{(1)}|}{2\lambda_0}\right) = \frac{\theta}{2}\left(1 \pm i\frac{\Delta\omega}{2\omega}\right)$$

Therefore, the averaging method proposed enables all wave propagation effects in periodically inhomogeneous media to be described.

A generalization of the model considered is a medium whose parameters are represented in the form

$$\lambda(x) = \lambda_0 + \sum_{m=1}^{\infty}\lambda_{1m}\cos m\theta x + \lambda_{2m}\sin m\theta x = \sum_{m=-\infty}^{\infty}\lambda^{(m)}\exp(-im\theta x) \tag{4.14}$$

Then under the same assumptions regarding the probabilistic properties of λ_{im}, the solution of (4.1) is represented, as above, in the form

$$u(x) = \sum_{m=-\infty}^{\infty} u(q - m\theta) \exp[-i(q - m\theta)x] \qquad (4.15)$$

where $u(q - m\theta)$ is found from the solution of the appropriate problem in the zero-th and first approximations. Its forbidden band of width $(\Delta\omega)^{(m)} = \omega \mid \lambda^{(m)} \mid \lambda_0^{-1}$ will be associated with each effective Fourier coefficient $\lambda^{(m)}$ in expansion (4.14). For $m = 1$ these bands are called higher-order forbidden bands /11/.

The expansion (4.14) can be used to solve the problem of the propagation of elastic waves in a periodically laminar two-component medium.

In this case $\lambda(x) = \lambda(x + 2l), \lambda(x) = \lambda_1$ for $-l < x < l_1, \lambda(x) = \lambda_2$ for $l_1 < x < l$ and can be expanded in a series (4.14) where $\lambda_0 = \langle\lambda\rangle = \lambda_1 c_1 + \lambda_2 c_2$, c_i is the bulk concentration of the i-th component, and $\lambda_{1m} = (\lambda_1 - \lambda_2)(m\pi)^{-1} \sin m\pi c_1$, $\lambda_{2m} = (\lambda_1 - \lambda_2)(m\pi)^{-1}[(-1)^m - \cos m\pi c_1]$, $\theta_m = m\pi l^{-1}$.

To find the zero-th approximation $\langle u\rangle$ we again assume the coefficients $\lambda_{1m}, \lambda_{2m}$ to be random and to possess the properties listed earlier, which requires the assumption of the randomness of $\lambda_1 - \lambda_2, l_1$. In this case the solution of (4.1) will have the form (4.15).

It follows from the above that the averaging method permits uniform solution of the wave propagation problem in media with a range of structure variation from completely disordered to periodically laminar. The effects of multiple scattering, velocity dispersion, and wave damping are taken into account here in the zero-th approximation. The first approximation makes the structure specific for the scattered field propagating in a medium with a given form of the elastic and density properties.

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